Introduction to Kajiwara–Payne Tropicalization and Berkovich Analytic Spaces

Sources:

- ↳ Sam Payne "Analytification is the limit of all tropicalization" ArXiV: 0805.1916
- ↳ Maclagan - Sturmfels
- ↳ Matt Baker's notes from the 2007 Arizona Winter School
- ↳ "Preliminaries" section of my paper with Philipp Jell "Construction of fully-faithful tropicalizations for curves in ambient dimension 3" ArXiV: 1912.02648

Berkovich

Background Algebra

An absolute value on a field K is a map $|\cdot| : K \to \mathbb{R}_{\geq 0}$ such that:

- ↳ $|0| = 0$
- ↳ $|\cdot| : (K^\times, 1, \cdot) \longrightarrow (\mathbb{R}_{>0}, 1, \cdot)$ is a group homomorphism (i.e. $|1| = 1$ and $|fg| = |f||g|$)
- ↳ triangle inequality: $|f+g| \leq |f| + |g|$

The absolute value is non-Archimedean if

$$|f+g| \leq \max\{|f|, |g|\}$$

Note: non-Arch. $\Updownarrow$ $\mathbb{Z}$ is bounded

Examples:

Archimedean

① if K is a number field (finite extension of $\mathbb{Q}$) then any embedding $K \hookrightarrow \mathbb{R}$ or $\mathbb{C}$ gives an absolute value by restricting the usual abs val. on $\mathbb{C}$.

non-Archimedean

② for every prime $p \in \mathbb{N}$, define $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ by

$$\left|p^n \frac{a}{b}\right|_p = \left(\tfrac{1}{p}\right)^n \quad \text{if } p \nmid ab$$

$\frac{1}{p}$ can be any positive real number < 1

③ More generally, if X is a suitably nice scheme or variety and D is a Weil divisor then we can define

$$|\cdot|_D : k(X) \to \mathbb{R}_{\geq 0}$$

by $\quad |f|_D = e^{-\mathrm{ord}_D(f)}$

② is a special case where $X = \operatorname{spec} \mathbb{Z}$, $D = (p)$

④ Hahn Series let k be a field, $\Gamma \leq \mathbb{R}$ an ordered subgroup

$$k[[t^\Gamma]] := \left\{ \sum_{\gamma \in I} c_\gamma t^\gamma : c_\gamma \in k,\ I \subseteq \Gamma \text{ is well-ordered} \right\}$$

$$\mathrm{val}\left(\sum_{\gamma \in I} c_\gamma t^\gamma\right) = \min\{\gamma : c_\gamma \neq 0\}$$

$$|f| = e^{-\mathrm{val}(f)}$$

eg. $\Gamma = \mathbb{Z} \rightsquigarrow$ Laurent series $\quad k((t))$

$\Gamma = \mathbb{Q} \rightsquigarrow$ Puiseux series $\quad k\{\{t\}\}$

If $k = k^{alg}$, char $k = 0$ then $k((t))^{alg} = k\{\{t\}\}$

To every non-Arch. valued field K, we associate several objects

$\mathrm{val}(f) = -\log|f|$ $\rightsquigarrow$ $\mathrm{val}(fg) = \mathrm{val}(f) + \mathrm{val}(g)$

$\rightsquigarrow$ $\mathrm{val}(f+g) \geq \min\{\mathrm{val}(f), \mathrm{val}(g)\}$

$\mathcal{O}_K = \{f : |f| \leq 1\} = \{f : \mathrm{val}(f) \geq 0\}$ Valuation ring

$m_K = \{f : |f| < 1\} = \{f : \mathrm{val}(f) > 0\}$ Maximal $\mathcal{O}_K$-ideal

$k = \mathcal{O}_K / m_K$ Residue field Eg. $K = \mathbb{Q}_p$, $\mathcal{O}_K = \mathbb{Z}_p$
$m_K = p\mathbb{Z}_p$, $k = \mathbb{F}_p$

Γ = image of $\mathrm{val} : K^\times \to \mathbb{R}$ Value group

$\Gamma \cong \mathbb{Z} \leftrightarrow$ Discrete valuation rings
eg from Weil divisors

Tropicalization

K a valued field with dense value group (in standard $\mathbb{R}$ topology) ($K = \overline{K}$ implies this when $|\cdot|$ is non-trivial)

$$\mathrm{trop} : \mathbb{A}^n_K \longrightarrow (\mathbb{R} \cup \{\infty\})^n$$

$$(a_1, \dots, a_n) \mapsto (\mathrm{val}(a_1), \dots, \mathrm{val}(a_n))$$

$\mathrm{Trop}(X)$ = closure of $\mathrm{trop}(X)$ in $\mathbb{R} \cup \{\infty\}$ topology

Example: $X = V(x+y+1)$ over $\mathbb{C}\{\{t\}\}$ (rational powers of t)

Key property: if $\mathrm{val}(f) \neq \mathrm{val}(g)$ then $\mathrm{val}(f+g) = \min\{\mathrm{val}(f), \mathrm{val}(g)\}$

So $\mathrm{val}(x+y+1) = \mathrm{val}(0) = \infty$ must mean the minimum is achieved at least twice.

$\hookrightarrow$ Case 1: $\mathrm{val}(x) = \mathrm{val}(y) \leq \mathrm{val}(1) = 0$

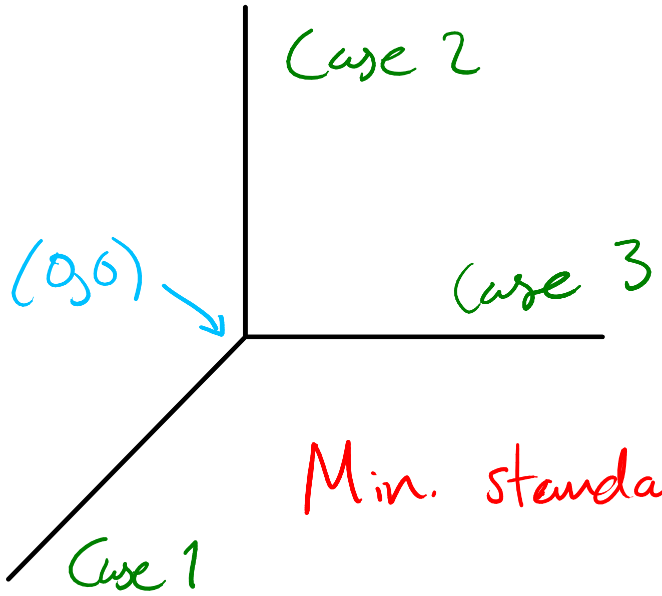
eg. $x = t^a$, $y = -t^a - 1$ $\rightsquigarrow$ $\mathrm{trop}(x,y) = (a,a)$
$a \leq 0$

$\hookrightarrow$ Case 2: $\mathrm{val}(x) = \mathrm{val}(1) \leq \mathrm{val}(y)$

eg. $x = -1 - t^a$, $y = t^a$ $\rightsquigarrow$ $\mathrm{trop}(x,y) = (0,a)$
$a \geq 0$

$\hookrightarrow$ Case 3: $\mathrm{val}(y) = \mathrm{val}(1) \leq \mathrm{val}(x)$ Same as 2.

Trop(X)

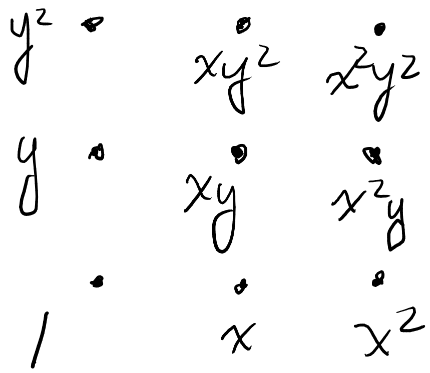


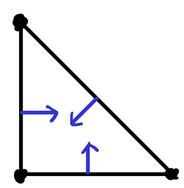
Min. standard tropical line

Tropical Hypersurfaces:

To find the shape of Trop(X) where $X = V(f) \subseteq (K^\times)^n$, we can look at the Newton polygon of f.

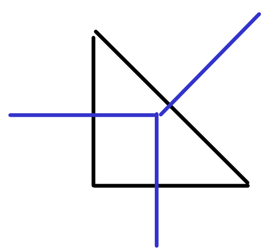
Eg. $f = x + y + 1$





Trop f is obtained from the inner normals

Or we can take the outer normals and rotate $180°$



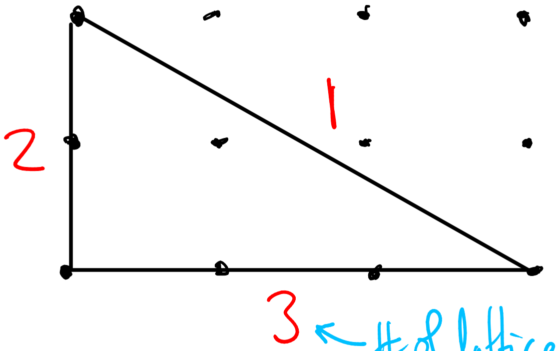
(Can be easier to visualize)

With a trivial valuation:

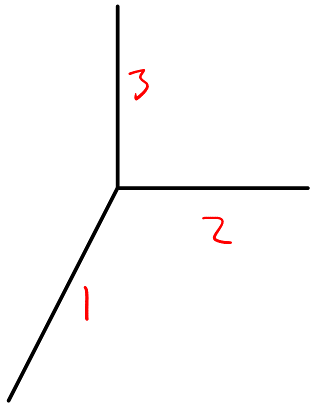
$$\mathrm{Newt}(f) = \mathrm{Conv}(\mathrm{Supp}(f))$$
$$= \mathrm{Conv}(\{(a_1, \ldots, a_n) : \text{the coefficient of } x_1^{a_1} \cdots x_n^{a_n} \text{ in } f \text{ is non-zero}\})$$

In this case, $\mathrm{Trop}(f)$ is a balanced, rational, polyhedral fan.

Eg



← # of lattice points on that edge



$$\sum \mathrm{mult}_i \times \mathrm{vec}_i = 0$$

$$1\boxed{(-3,-2)} + 2(0,1) + 3(1,0) = 0$$

generator of $(\text{ray} \cap \mathbb{Z}^n)$

With a non-trivial valuation:

↳ $f = \sum \{c_{\underline{a}} x^{\underline{a}} : \underline{a} \in \mathbb{Z}^n\}$

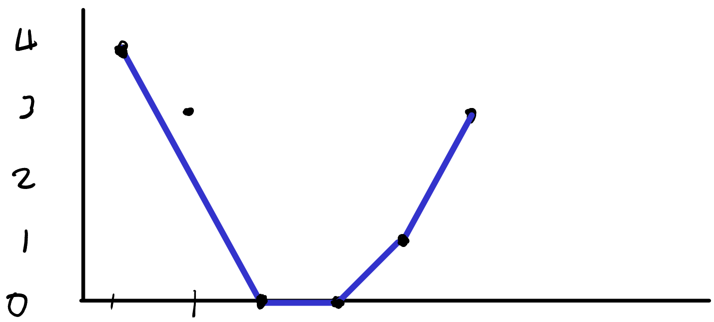
↳ Valuated Support $= \{(\underline{a}, \mathrm{val}(c_{\underline{a}})) : \underline{a} \in \mathbb{Z}^n\} \subseteq \mathbb{Z}^{n+1}$

↳ Take the convex hull

↳ Take the 'lower' faces (normal vector has last coordinate >0)

↳ Project the $n-1$ skeleton to $\mathbb{R}^n = \{x_{n+1} = 0\}$

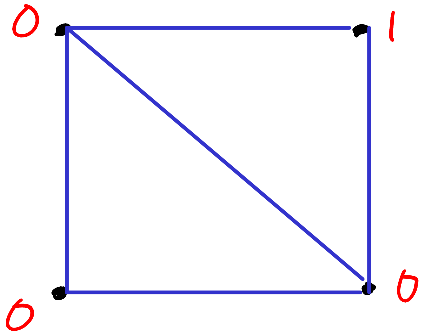
Eg. $f = t^4 + t^3 x + x^2 + x^3 + t x^4 + t^3 x^5$



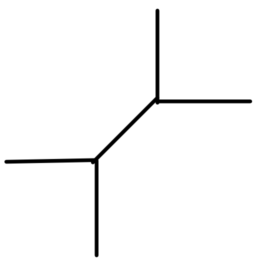
$\downarrow$ Project to $x_2 = 0$

Eg Tropical quadratic

$f = 1 + x + y + txy$



$\overset{\mathrm{Trop}(f)}{\rightsquigarrow}$



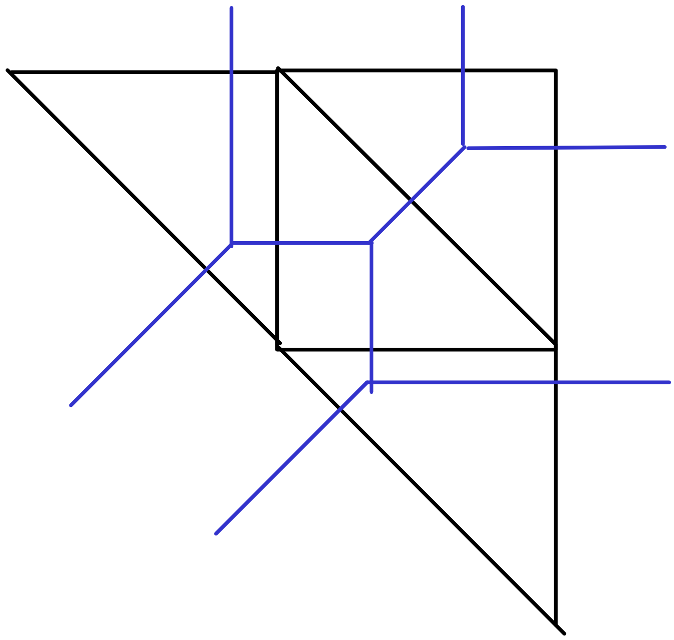
In general, Trop(V(f)) is a balanced, rational, polyhedral complex.

Tropical variety

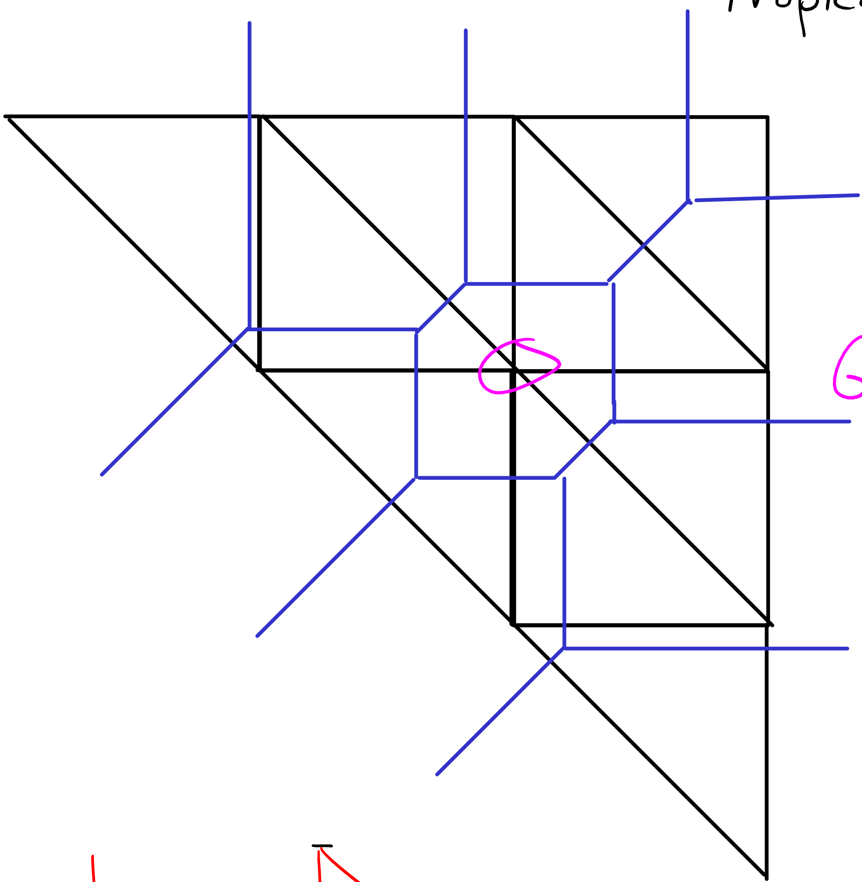
Tropicalized variety

More Pictures:

Projective degree 2 tropical curve

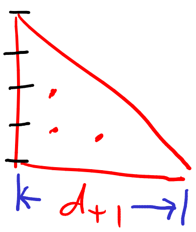


Projective degree 3 tropical curve



Genus = # internal vertices

# of internal vertices:



$1+2+\cdots+(d-2) = \binom{d-1}{2}$

If $X = V(I) \subseteq (K^\times)^n$ then

$$\mathrm{Trop}(X) = \bigcap_{f \in I} \mathrm{Trop}(V(f))$$

TFAE (Alg. closed, non-trivial valuation)

1. $\displaystyle\bigcap_{f \in I} \mathrm{Trop}(V(f))$

2. closure of $\mathrm{trop}(X) = \{(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) : \underline{x} \in X\}$

3. $\{\underline{w} \in \mathbb{R}^n : \mathrm{in}_{\underline{w}}(I) \not\ni \text{a monomial}\}$

Think "Sum of all terms where $\mathrm{val}(c_{\underline{a}}) + \langle \underline{a}, \underline{w} \rangle$ is minimum"

Formally: $W = \min\{\mathrm{val}(c_{\underline{a}}) + \langle \underline{a}, \underline{w} \rangle\}$

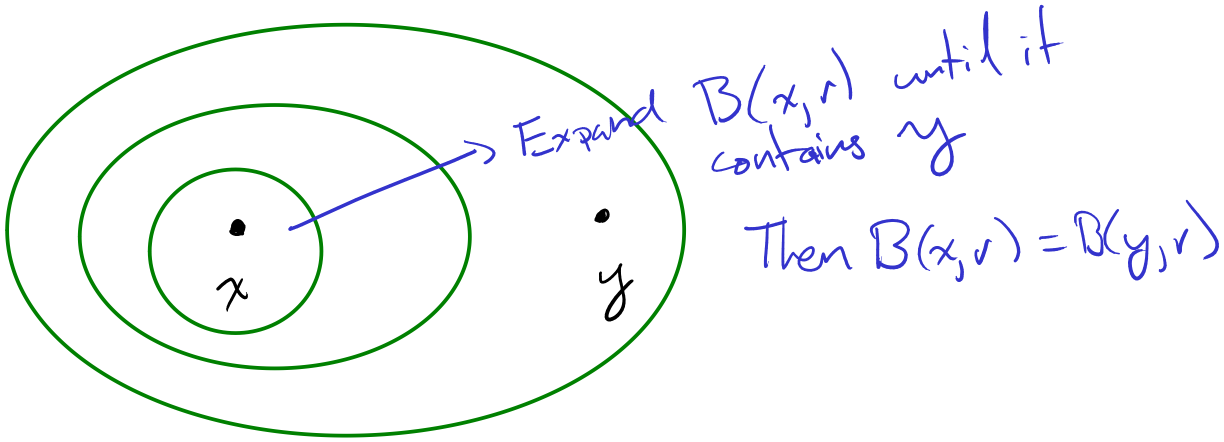
$$\mathrm{in}_{\underline{w}}(f) = \sum_{\mathrm{val}(c_{\underline{a}}) + \langle \underline{a}, \underline{w} \rangle = W} \overline{c_{\underline{a}}\, t^{-\mathrm{val}(c_{\underline{a}})}}\, \underline{x}^{\underline{a}} \in k[\underline{x}]$$

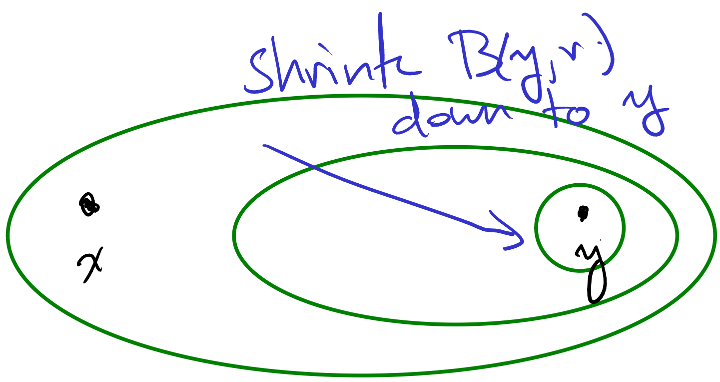
Berkovich Analytic Spaces

Idea: to have analytic geometry for non-Archimedean fields
Refine Tate's rigid analytic spaces

$|\cdot|$ induces a Hausdorff topology on K but it's totally disconnected in the non-Arch. setting, nor is it locally compact.

We get around this by adding the following kinds of paths to K:





So what we need to do is add these balls as points.

↳ now the space is path-connected
↳ we will worry about compactness later

A multiplicative seminorm on an algebra $\mathcal{A}$ is a function $\|\cdot\| : \mathcal{A} \to \mathbb{R}_{\geq 0}$ such that

- $\|0\| = 0$ and $\|1\| = 1$
- $\|fg\| = \|f\| \, \|g\|$
- $\|f+g\| \leq \|f\| + \|g\|$

The Berkovich analytification of a variety X is

$$X^{an} := \{ (x, |\cdot|_x) : x \in X \text{ and } |\cdot|_x \text{ is a norm on } k(x) \text{ extending } |\cdot|_k \}$$

The topology is the initial topology with respect to:

$$\begin{array}{ccc} X^{an} & \longrightarrow & X \\ (x, |\cdot|_x) & \longmapsto & x \end{array} \quad \text{and} \quad \begin{array}{ccc} U^{an} & \longrightarrow & \mathbb{R} \\ (x, |\cdot|_x) & \longmapsto & |f(x)|_x \end{array}$$

$\forall U$ open, $f \in \mathcal{O}_X(U)^\times$

$\mathrm{Spec}(R)^{an} = \{ \text{seminorms} : R \to \mathbb{R}_{\geq 0} \text{ that extend } |\cdot|_k \}$

Coarsest topology such that

$$\begin{array}{rccc} ev_a : & \mathrm{Spec}(R)^{an} & \longrightarrow & \mathbb{R}_{\geq 0} \\ & |\cdot| & \longmapsto & |a| \end{array}$$

is continuous for all $a \in R$.

Compact-open topology as a subset of $\mathrm{Fun}(R, \mathbb{R}_{\geq 0})$

discrete

Example $\mathbb{A}^{1,an}$ and $\mathbb{P}^{1,an}$ over an algebraically closed field

Case 1: x is closed $\Rightarrow k(x) = K \Rightarrow |\cdot|_x = |\cdot|_K$ } Type I points

So $\underbrace{X(K) \hookrightarrow X^{an}}_{\text{general phenomenon}}$

Case 2: x is the generic point $\Rightarrow k(x) = K(t)$

Question: what kinds of norms are there on $K(t)$?

Type II and III points: if $f = \sum c_i t^i \in K[t]$:

$$|f|_{a,r} = \sup_{|z-a|_K \le r} |f(z)|_K = \max\{|c_i|_K r^i\}$$

Maximum Modulus Principle

Type II if $r \in |K^\times|$, type III otherwise.

Berkovich Classification Theorem

Every point of $\mathbb{A}^{1,an}$ corresponds to a nested sequence of closed balls:

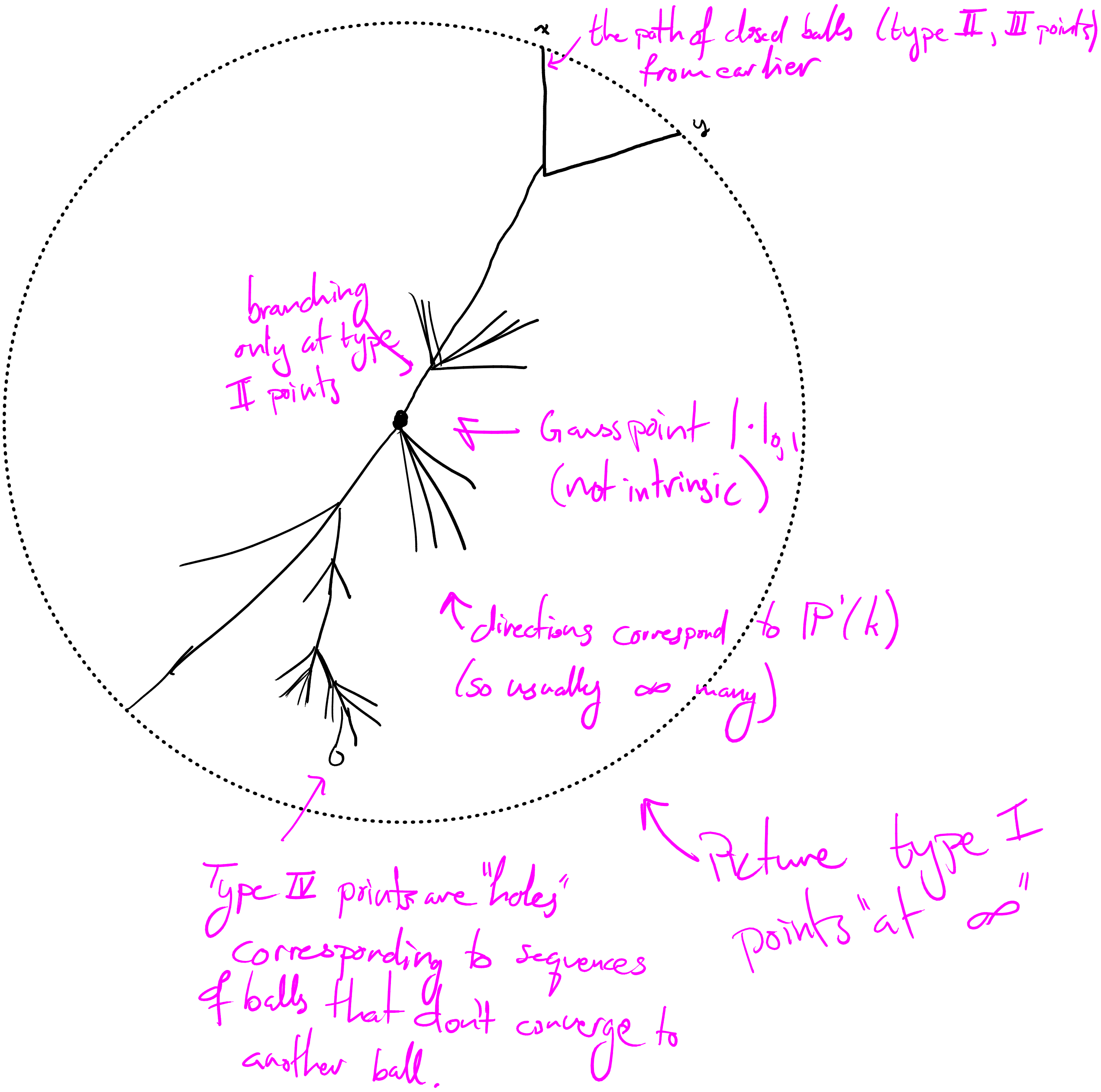
$$B(a_1, r_1) \supseteq B(a_2, r_2) \supseteq \cdots$$

and $|f|_x = \lim |f|_{a_i, r_i}$

Type IV points are every point that is not type I – III.

↳ need these for compactness

$\mathbb{P}^{1,an}$:



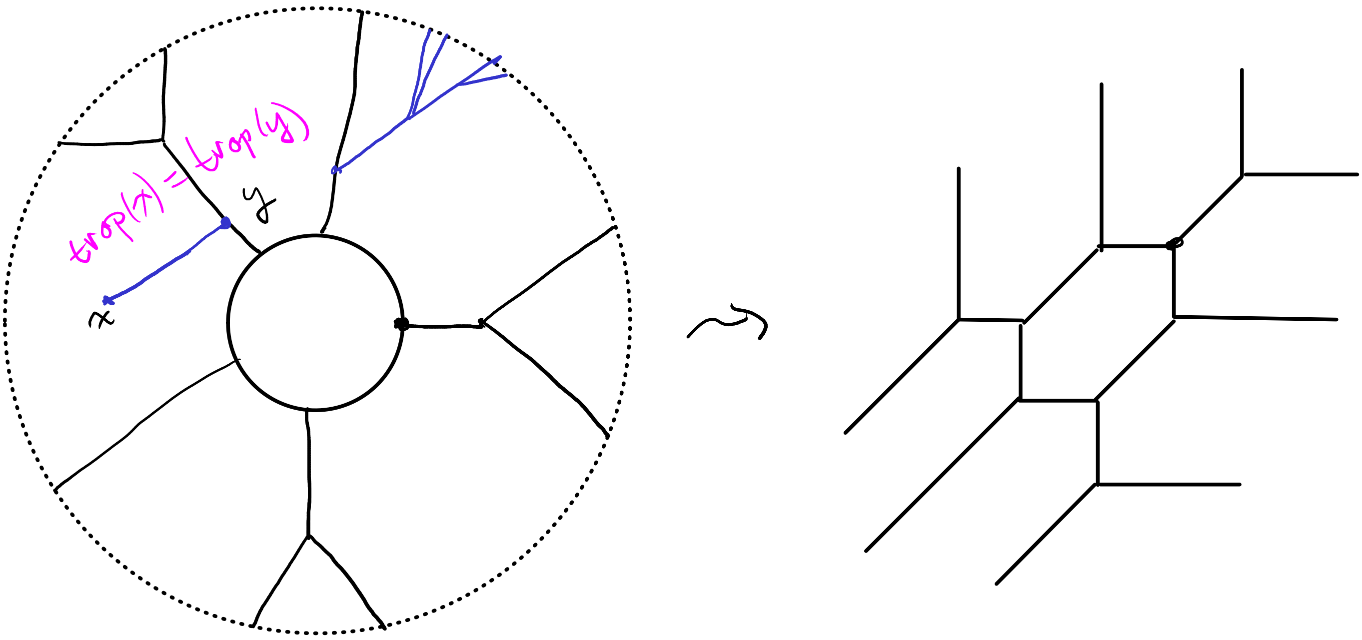
$\mathbb{P}^{1,an}$ is compact, Hausdorff, path-connected, almost a metric space:

Length of the path from $|\cdot|_{a,r}$ to $|\cdot|_{a,s}$ is

$$|\log(r) - \log(s)|.$$

So Type I points are ∞ far away from type II–IV points

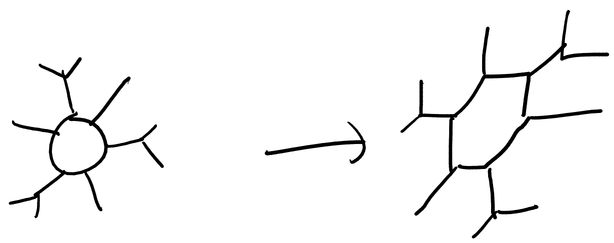
Tropicalizing an Elliptic Curve by Example

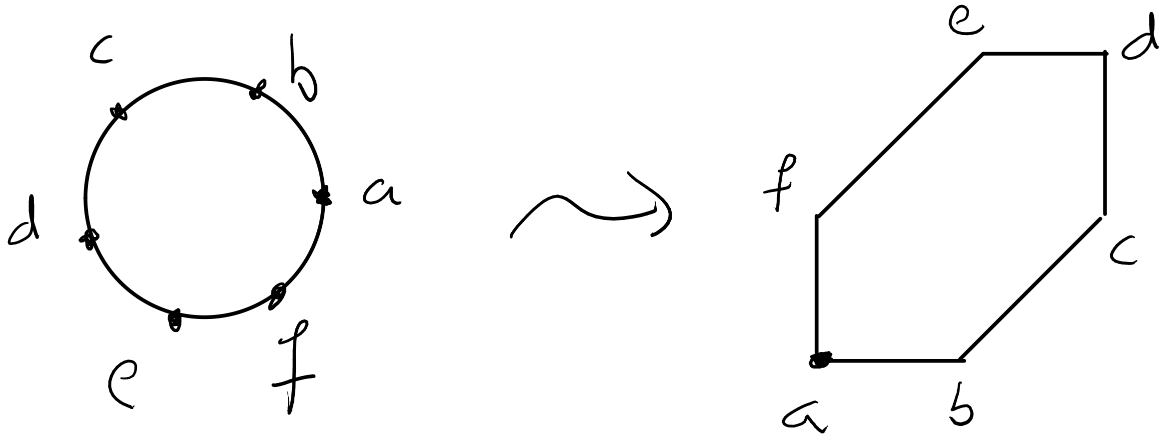


See example 3.5 of Jell "Constructing smooth and fully faithful tropicalizations for Mumford curves."

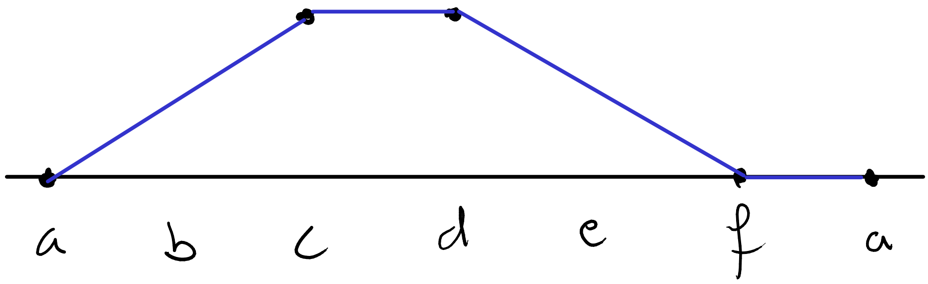
ArXiV: 1805.11594

The question here is: given piecewise linear functions from the skeleton to Trop(X), can we lift them to functions on X.

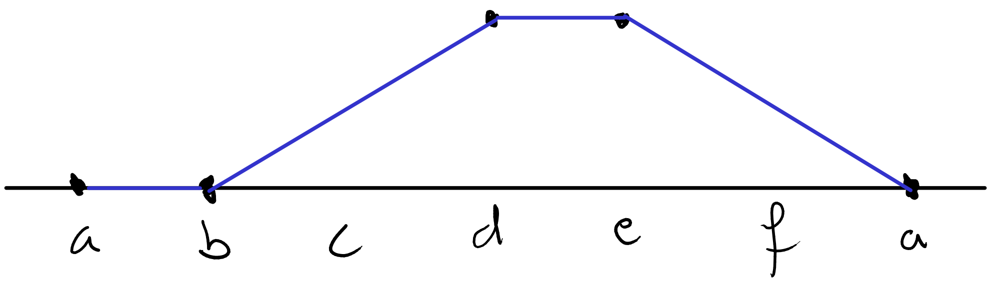
F:  , $F = -\log|f|$?



F_x:



F_y:



Can talk about div(F_x), div(F_y). Leads to a whole theory of tropical divisors.

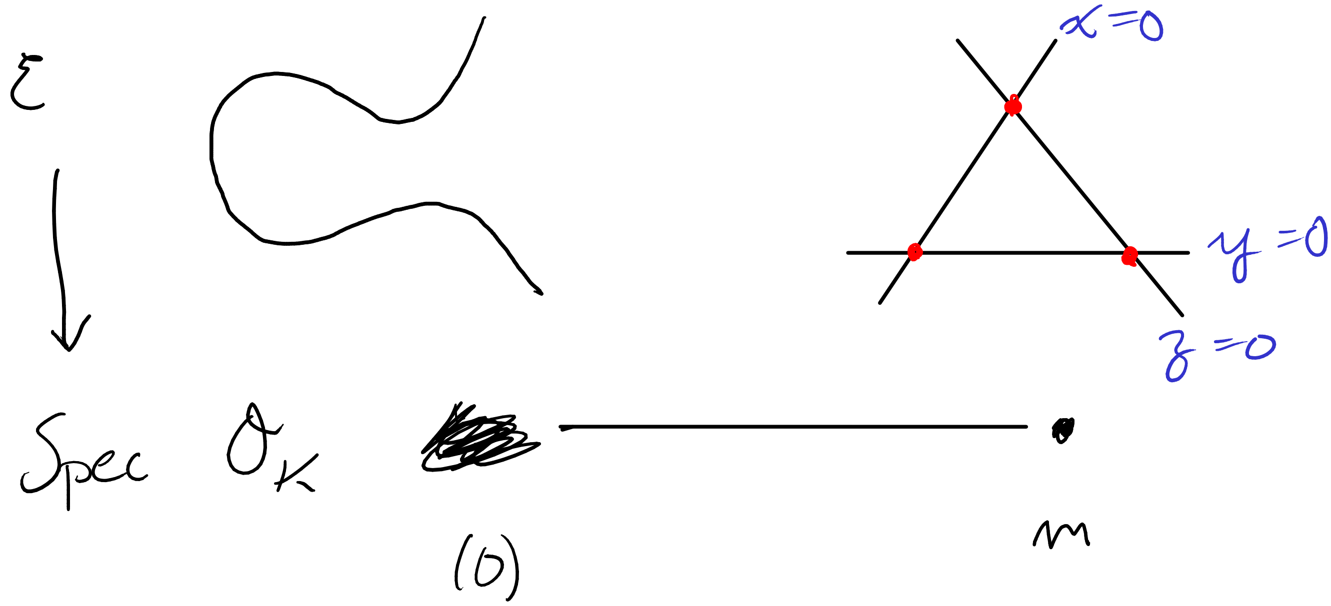
A rough idea of why E^{an} looks like this

Consider $f = xyz + t^n(x^3+y^3+z^3)$ over $\mathcal{O}_K$

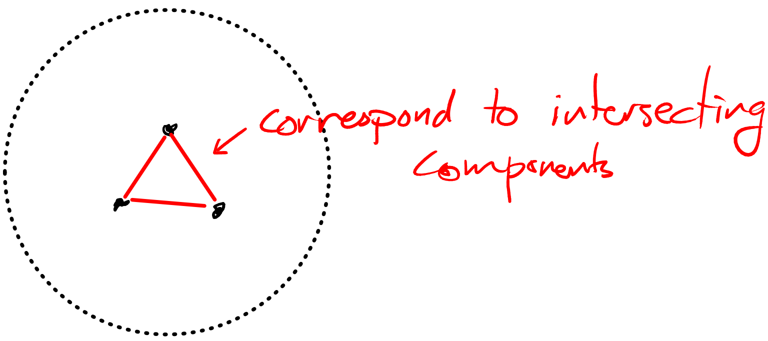
$$\downarrow \text{mod } m_K$$

$$xyz$$

$$\mathcal{E} = V(f) \subseteq \mathbb{P}^2_{\mathcal{O}_K} \to \operatorname{Spec} \mathcal{O}_K$$



Each irreducible component of $\mathcal{E}_m$ is a Weil divisor which gives a distinguished point of E^{an}.



Tropical Projective Space

Classically we can obtain $\mathbb{P}^1$ in two ways:

By gluing

$$\mathbb{A}^1 \longrightarrow \mathbb{A}^1$$
$$x \longmapsto x^{-1}$$

As a quotient

$$\mathbb{A}^2 \setminus (0,0) \, / \, \text{torus } \mathbb{G}_m^1$$

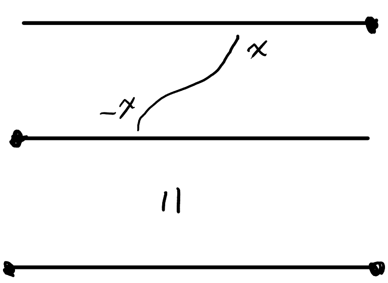
$\Big\downarrow$ tropically

$$\mathbb{A}^1 \longrightarrow \mathbb{A}^1$$
$$x \longmapsto -x$$

tropical $\frac{1}{x}$

$$(\overline{\mathbb{R}})^2 \setminus (\infty, \infty) \, / \, \text{tropical torus}$$

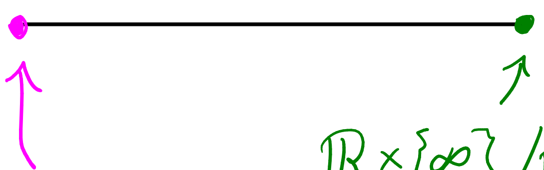
$$(a,b) \sim \lambda \odot (a,b) = (a+\lambda, b+\lambda)$$



$\mathbb{A}^1 = \mathbb{R} \cup \{\infty\}$

$\mathbb{P}^1 = \mathbb{R} \cup \{\pm\infty\}$

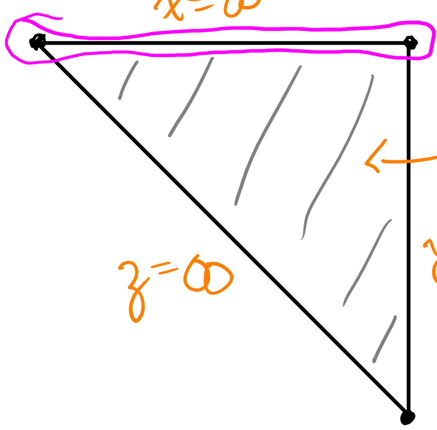
$\mathbb{R}^2 / \mathbb{R}\cdot\mathbb{1}$



$\{\infty\} \times \mathbb{R} / \mathbb{R}\cdot\mathbb{1}$

$\mathbb{R} \times \{\infty\} / \mathbb{R}\cdot\mathbb{1}$

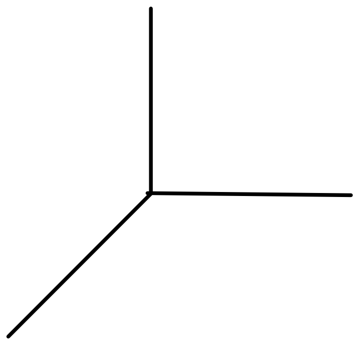
$\mathbb{P}^2$



In general tropical $\mathbb{P}^n$ is an n-dimensional simplex

The projective tropical line

Recall: $x+y+1=0$ in $\mathbb{G}_m^2$:



There are 3-additional points of infinity

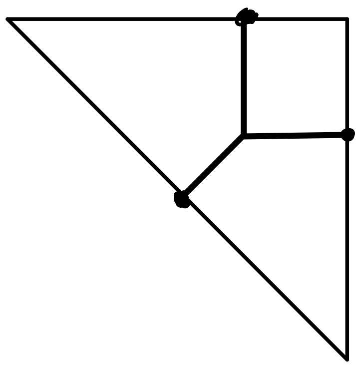
↳ $\mathrm{trop}(-1,0) = (0,\infty)$

↳ $\mathrm{trop}(0,-1) = (\infty, 0)$

↳ $[x:y:z]$ with $x+y+z=0$ at $z=0$

$[x:-x:0] \overset{\mathrm{trop}}{\rightsquigarrow} [r:r:\infty]$ $(\mathrm{val}(x) = \mathrm{val}(-x))$

$= [0:0:\infty]$



Tropicalization of Toric Varieties

Usual set-up:

$N \cong \mathbb{Z}^n$ lattice, $N_{\mathbb{R}} := N \otimes \mathbb{R} \cong \mathbb{R}^n$

$M = \mathrm{Hom}(N, \mathbb{Z})$ dual lattice

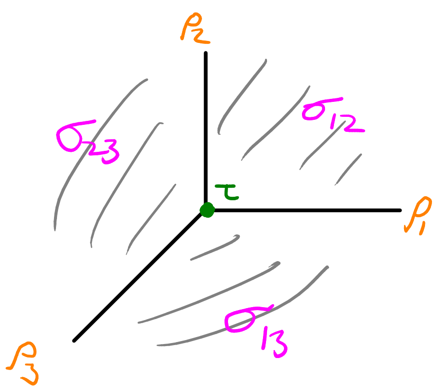
Δ a fan in $N_{\mathbb{R}}$, $Y = Y(\Delta)$ toric variety

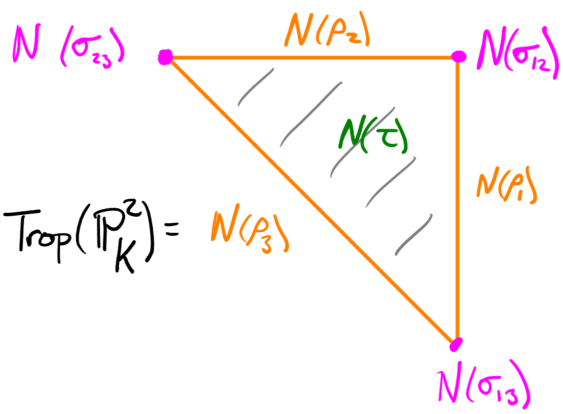
↳ For each cone $\sigma \in \Delta$, let

$$N(\sigma) = N_{\mathbb{R}} / \mathrm{span}(\sigma)$$

↳ Define $\mathrm{Trop}(Y) = \bigsqcup_{\sigma} N(\sigma)$

Example: $\mathbb{P}^2_K = Y(\Delta)$ where

$\Delta =$ 

$\rightsquigarrow \mathrm{Trop}(\mathbb{P}^2_K) =$ 

Glueing the $N(\sigma)$'s

$\{f \in M;\ \langle f, x\rangle \geq 0, \forall x \in \sigma\}$

Recall $\sigma \longleftrightarrow K[A_\sigma]$, $A_\sigma = \sigma^\vee \cap M$

$U_\sigma = \mathrm{Spec}(K[A_\sigma])$

$\mathbb{R}$
G

Question: if $\mathrm{Trop}(\mathbb{A}^1_K) = \mathrm{Trop}(\mathrm{Spec}(K[\mathbb{N}])) = \overline{\mathbb{R}} =: \mathbb{R} \cup \{\infty\}$
and $\mathrm{Trop}(\mathbb{G}^1_{m,K}) = \mathrm{Trop}(\mathrm{Spec}(K[\mathbb{Z}])) = \mathbb{R}$ (also with $\mathbb{R}$-action)

What should $\mathrm{Trop}(\mathrm{Spec}(K[S]))$ be?

$\rightsquigarrow$ Define $\mathrm{Trop}(\mathrm{Spec}(K[S])) = \mathrm{Hom}_{\text{semigrp}}(S, \overline{\mathbb{R}})$ $\swarrow$ with $+$

We give $\mathrm{Hom}(S, \overline{\mathbb{R}})$ the compact-open topology,

$\uparrow$ $f_n \to f$ iff $f_n(x) \to f(x)$ $\forall x$

E.g. $\sigma =$ origin in fan$(\mathbb{P}^2_K)$

$$A_\sigma \cong \mathbb{Z}^2 \text{ and } \mathrm{Hom}(\mathbb{Z}^2, \overline{\mathbb{R}}) \cong (\overline{\mathbb{R}}^\times)^2 = \mathbb{R}^2$$

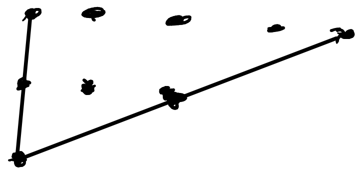
$\sigma = \sigma_{12}$  $= \sigma^\vee$

$$A_\sigma = \mathbb{F}_1[x, y] \cong \mathbb{N}^2 \text{ and } \mathrm{Hom}(\mathbb{N}^2, \overline{\mathbb{R}}) \cong \overline{\mathbb{R}}^2$$

Point: if $\tau \preceq \sigma$ then $\mathrm{Trop}(U_\tau) \subseteq \mathrm{Trop}(U_\sigma)$.

Because $\sigma^\vee \cap M \subseteq \tau^\vee \cap M$

$$\begin{tikzcd} \sigma^\vee \cap M \arrow[dr] & & \tau^\vee \cap M \arrow[dl] \arrow[ll, phantom, "\xleftarrow{\text{restricts}}"] \\ & \overline{\mathbb{R}} & \end{tikzcd}$$

E.g. $\sigma^\vee = \{(1,0),(1,2)\}$ 

$$\Rightarrow A_\sigma = \langle (1,0),(1,1),(1,2)\rangle$$

$$\Rightarrow \mathrm{Hom}(A_\sigma, \overline{\mathbb{R}}) = \{(a,b,c) \in \overline{\mathbb{R}} : a+c=2b\}$$

Point: because A_σ is finitely generated, $\mathrm{Hom}(A_\sigma, \overline{\mathbb{R}})$ is a subspace of $\overline{\mathbb{R}}^m$ (same topology).

Remark: If $\phi \in \mathrm{Trop}(U_\sigma)$ then

$$\phi(u) = \infty \Rightarrow \phi(u+v) = \infty \quad \forall v$$

$$\text{if } \phi(u+v) = \infty \Rightarrow \phi(u) = \infty \text{ or } \phi(v) = \infty$$

So $\{u : \phi(u) \neq \infty\}$ is a face of $\sigma^\vee$.

$$\updownarrow$$

$\tau^\perp$ is a group

$$\mathrm{Hom}(\tau^\perp \cap M, \overline{\mathbb{R}}) = \mathrm{Hom}(\tau^\perp \cap M, \mathbb{R})$$

$$= \mathrm{Hom}(\{f : N \to \mathbb{Z} \mid \langle f, x\rangle = 0 \ \forall x \in \tau\}, \mathbb{R})$$

$$\cong N \otimes \mathbb{R} / \mathrm{span}(\tau) = N(\tau)$$

So we have a filtration into toric orbits:

$$\mathrm{Trop}(U_\sigma) = \coprod_{\tau \preceq \sigma} N(\tau).$$

Toric varieties as a quotient (Cox, 1995, The Homogeneous Coordinate Ring of a Toric Variety)

For simplicity, assume Δ is smooth $\Leftrightarrow$ the generators of any $\sigma \in \Delta$ can be extended to a basis.

<u>Cox ring</u>: Recall that toric-invariant Weil divisors correspond to $\Delta(1)$ (1-skeleton)

Say $\Delta(1) = \{\rho_1, \dots, \rho_s\}$ where $\rho_i \leftrightarrow D_i$

Define $S = K[x_1, \dots, x_s]$ where $\deg(x_i) = [D_i] \in Cl(Y)$

The graded ring S is called the Cox ring of Y.

Let v_i be the smallest integer vector spanning ρ_i. Define

$$\begin{aligned} M &\xrightarrow{V} \mathbb{Z}^{\Delta(1)} \\ u &\longmapsto \sum_i \langle u, v_i \rangle \cdot \rho_i \end{aligned}$$

We have an exact sequence

Torsion free b/c smooth

$$\begin{array}{c} \mathbb{Z}^n \\ \cong \\ 0 \to M \xrightarrow{V} \mathbb{Z}^{\Delta(1)} \longrightarrow Cl(Y) \to 0 \end{array}$$

$\mathrm{Hom}(-, K^\times)$

$$T^n \xleftarrow{V^T} (K^\times)^s \longleftarrow H \longleftarrow 0$$

torus of Y Torus

This sequence gives $H \hookrightarrow (K^\times)^s \circlearrowleft \mathbb{A}^s_K$

We have $\mathcal{Y} \cong \left(\mathbb{A}^s_K \setminus V(B)\right)/H$ for some ideal B.

Eg. $\mathcal{Y} = \mathbb{P}^n_K$, $\Delta(1) = \{\operatorname{span}(e_i) : i=1,\dots,n\}$
$\cup \{\operatorname{span}(-\sum e_i)\}$

So $\operatorname{Cox}(\mathbb{P}^n_K) = K[x_0,\dots,x_n]$

and $[D_i] = [D_j]$ $\forall i,j$ so

it's the usual grading.

$$\mathbb{Z}^{n+1} \xrightarrow{\deg} \mathbb{Z} = \mathrm{Cl}(\mathbb{P}^n)$$

$$\begin{aligned} (K^\times)^{n+1} &\longleftarrow K^\times \\ (\lambda,\dots,\lambda) &\longmapsfrom \lambda \end{aligned}$$

So $\mathbb{P}^n_K = \left(\mathbb{A}^{n+1} \setminus (0,\dots,0)\right)/K^\times$

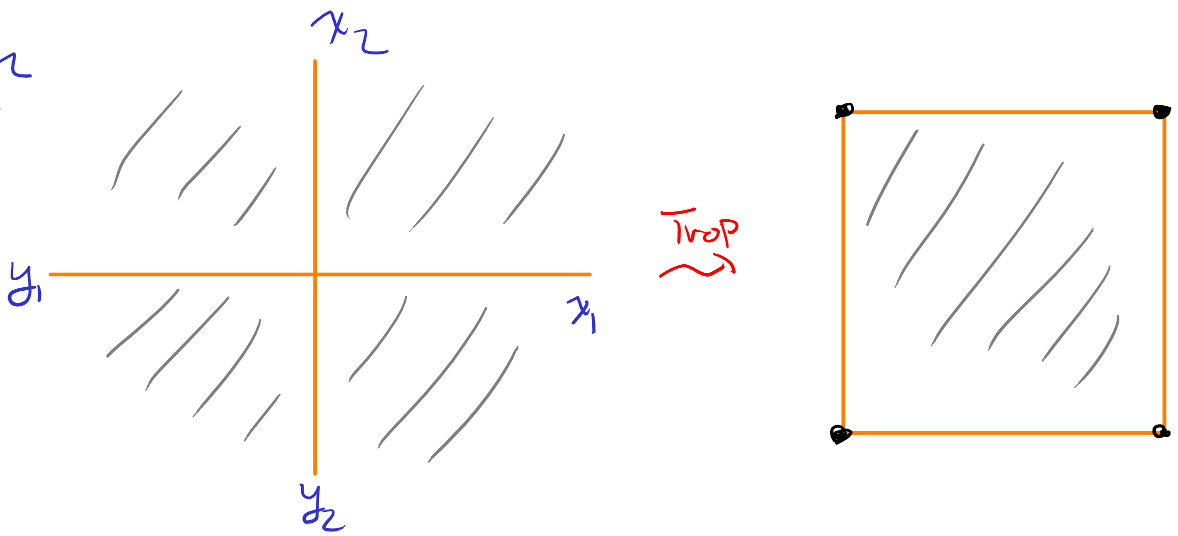
The ideal B is $\left\langle \prod_{v_i \notin \sigma} x_i : \sigma \in \Delta \right\rangle$

"Cox irrelevant ideal"

Point: the quotient presentation of $\mathbb{P}^n_K$ applies to general toric varieties. Also computable.

Similarly, $\mathrm{Trop}(\mathcal{Y}) \cong \big(\mathrm{Trop}(\mathbb{A}^5) \setminus \mathrm{Trop}(V(B))\big) / \mathrm{Trop}(H)$.

E.g. $(\mathbb{P}^1)^2$



$\mathrm{Cox}((\mathbb{P}^1)^2) = K[x_1, y_1, x_2, y_2]$

bigraded: $\deg(x_1) = \deg(y_1) = (1,0)$

$\deg(x_2) = \deg(y_2) = (0,1)$

$B = \langle x_1 x_2, x_1 y_2, x_2 y_1, y_1 y_2 \rangle = \langle x_1, y_1 \rangle \cap \langle x_2, y_2 \rangle$

$\mathbb{A}^4 \setminus V(B) \cong (\mathbb{A}^2 \setminus (0,0))^2$

So $(\mathbb{P}^1_K)^2 = (\mathbb{A}^2 \setminus (0,0))^2 / \{(\lambda, \lambda, \mu, \mu)\}$

Analytic Toric Varieties and Closed Subvarieties

The maps
$$\begin{array}{cl} \mathrm{Hom}(A_\sigma, K) & = \mathrm{Hom}(K[A_\sigma], K) \\ \big\downarrow \mathrm{trop}_Y & = \mathcal{U}_\sigma(K) \\ \mathrm{Hom}(A_\sigma, \overline{\mathbb{R}}) & \end{array}$$

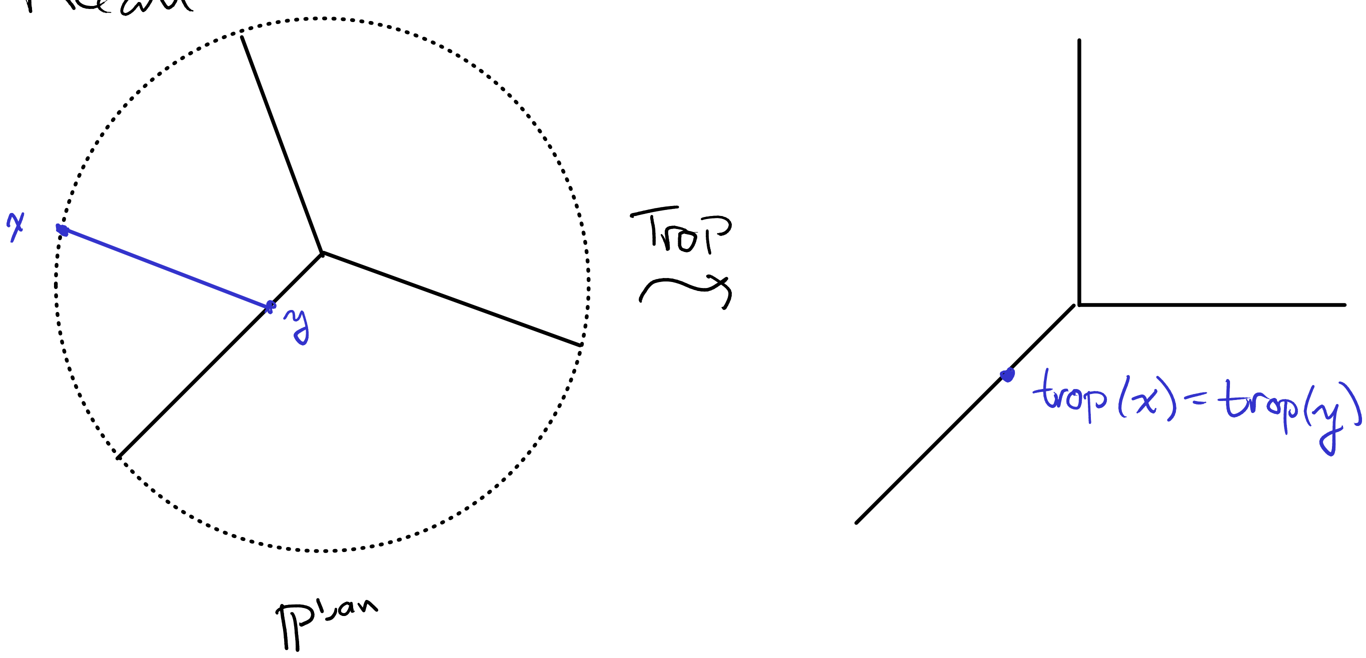
give a tropicalization map $\underbrace{Y(K) \longrightarrow \mathrm{Trop}(Y)}_{\text{functorial}}$

If $\iota : X \to Y(\Delta)$ is a closed subvariety then we define
$$\mathrm{Trop}(X, \iota) = \overline{\mathrm{trop}_Y(\iota(X(K)))}.$$

(K algebraically closed, $|\cdot|_K$ nontrivial)

Extending trop_Y to Y^{an}

Recall



The embedding $\iota : X \longrightarrow \mathcal{Y}(\Delta)$ gives an open cover of X:

$$X = \bigcup_{\sigma} X \cap U_\sigma$$

Which, in turn, gives

$$X^{an} = \bigcup_{\sigma} X^{an} \cap U_\sigma^{an}$$

and we obtain a map $\mathrm{trop} : X^{an} \longrightarrow \mathrm{Trop}(X, \iota)$ as follows:

If $\iota : X \cap U_\sigma \to U_\sigma \subseteq \mathbb{A}^n$ is given by $\iota(x) = (f_1(x), \dots, f_n(x))$ then

$$\mathrm{trop}_\sigma(|\cdot|_x) = (-\log|f_1|_x, \dots, -\log|f_n|_x)$$

Alternatively: $|\cdot|_x \in U_\sigma^{an}$ maps to $\phi_x \in \mathrm{Hom}(A_\sigma, \overline{\mathbb{R}})$ where

$\uparrow$ $\mathrm{Trop}(U_\sigma)$

$$\phi_x(\chi^m) = -\log|\chi^m|_x$$

$\uparrow$ $A_\sigma \subseteq M$

We glue these maps trop_σ to obtain $\mathrm{trop} : X^{an} \longrightarrow \mathrm{Trop}(\mathcal{Y})$.

General idea of Sam Payne's paper: (for affine varieties)

1. Every tropicalization map $\mathrm{trop}_\iota : X^{an} \to \mathrm{Trop}(\mathcal{Y})$ is continuous since, by definition, the topology on X^{an} is the coarsest s.t. every
$$\mathrm{ev}_f : |\cdot|_x \longmapsto |f|_x \qquad (f \in \mathcal{O}_X(U))$$
is continuous.

2. $\mathbb{P}^{1,an}_K \longrightarrow \mathrm{Trop}(\mathbb{P}^1)$ "$\mathbb{R} \cup \{\pm\infty\}$" is proper because $\mathbb{P}^{1,an}$ is compact, Hausdorff

3. Similarly, restricting to $\mathrm{trop}^{-1}(\mathrm{Trop}(\mathbb{A}^1)) = \mathrm{trop}^{-1}(\overline{\mathbb{R}}) = \mathbb{A}^{1,an}$ gives a proper map $\mathbb{A}^{1,an}_K \to \overline{\mathbb{R}}$.

4. So for every embedding $\imath : X \hookrightarrow \mathbb{A}^m$, the tropicalization

$$\mathrm{trop}_\imath : X^{an} \hookrightarrow \mathbb{A}^{m,an} \to \overline{\mathbb{R}}^m$$

is proper.

5. Since $X(K)$ is dense in X^{an} and $\mathrm{trop}_\imath$ is proper,

$$\mathrm{Trop}(X) := \overline{\mathrm{trop}(X(K))} = \mathrm{trop}_\imath(X^{an}).$$

6. By (1), $\pi : X^{an} \longrightarrow \varprojlim_{\imath : X \to \mathbb{A}^m} \mathrm{Trop}(X, \imath)$

is a homeomorphism $\Leftrightarrow$ it is bijective.

Similarly, if X is quasiprojective, then $X^{an} \cong \varprojlim \mathrm{Trop}(X, \imath)$ over all closed embeddings of X into a g.p. toric variety.

Proof of injectivity: Suppose $\pi(|\cdot|_x) = \pi(|\cdot|_{x'})$ then $|\cdot|_x = |\cdot|_{x'} \Leftrightarrow |f|_x = |f|_{x'}$ $\forall f \in K[X]$.

Extend $f_1 = f$ to a generating set $f_1, \ldots, f_n$ of $K[A]$ this gives an embedding

$$\imath = (f_1, \ldots, f_n) : X \longrightarrow \mathbb{A}^n$$

By hypothesis, $\leftarrow (-\log|f_1|_x, \dots, -\log|f_m|_x)$

$$\mathrm{trop}_\iota(|\cdot|_x) = \mathrm{trop}_\iota(|\cdot|_{x'})$$

Hence $|f|_x = |f|_{x'}$.

inj. +

Proof of surjectivity: let $(v_\iota) \in \varprojlim \mathrm{Trop}(X, \iota)$. We want to define $|\cdot|_x \in X^{an}$ with $\pi(|\cdot|_x) = (v_\iota)$.

To define $|\cdot|_x$, we need to say what $|f|_x$ is for every $f \in K[X]$.

$\hookrightarrow$ Choose an embedding $\iota = (f_1, \dots, f_m)$ with $f_1 = f$, then

$$|f|_x = \exp(-\text{first coord. of } v_\iota)$$

We need to check that this is well-defined:

$\hookrightarrow$ Suppose $j = (g_1 = f, \dots, g_n)$. Now look at

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d, "j"'] \arrow[dr, "\iota \times j"] & \mathbb{A}^m \arrow[d] \\ \mathbb{A}^n \arrow[r] & \mathbb{A}^{m+n} \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "\iota \times j"'] \arrow[dr, "\widetilde{\iota \times j}"] & \\ \mathbb{A}^{m+n} \arrow[rr] & & \mathbb{A}^{m+n} \end{tikzcd}$$

swap 1st and (m+1)st coords.

Since (v_ι) is an inverse system, $v_{\iota \times j} = v_{\widetilde{\iota \times j}} \Rightarrow |f_1|_x = |g_1|_x$.

Finally, to check that $|\cdot|_x$ is a seminorm, consider, for ex. embedding $\iota = (f_1 = f, f_2 = g, f_3 = fg, f_4, \dots, f_m)$. Then necessarily

$$(v_\iota)^3 = (v_\iota)^1 + (v_\iota)^2.$$

Similarly for Δ-inequality.

Surj. +

Tropical Semifield

$$\mathbb{T} = \left(\mathbb{R} \cup \{\infty\}, \underset{\min}{\underset{\shortparallel}{\oplus}}, \underset{+}{\underset{\shortparallel}{\odot}}\right) \qquad \begin{matrix} \infty = 0_{\mathbb{T}} \\ 0 = 1_{\mathbb{T}} \end{matrix}$$

No negatives:

$$x \oplus y = 0_{\mathbb{T}} \quad \text{iff} \quad x = y = \infty$$

Instead, we say $\bigoplus_{i=1}^{n} a_i$ tropically vanishes if $\min\{a_1, \dots, a_n\}$ occurs at least twice.

Equivalently:

$$\bigoplus_{i=1}^{n} a_i = \bigoplus_{i \neq j} a_i \qquad \forall j \in \{1, \dots, n\}$$

"Bend Relations"

Tropical Hyperfield

Let K be any valued field whose value group is $\mathbb{R}$

$$\mathrm{val} : K \twoheadrightarrow \mathbb{R} \cup \{\infty\}$$

hypersum

define $a \boxplus b = \{\mathrm{val}(f+g) : \mathrm{val}(f) = a, \mathrm{val}(g) = b\}$

$a \cdot b =$ "$\mathrm{val}(f \cdot g)$" $= a + b$

Intrinsic; doesn't depend on K

Eg. $v_p : \mathbb{Q}_p \twoheadrightarrow \mathbb{Z} \cup \{\infty\} \qquad (p \neq 2)$

If $\mathrm{val}(f) = \mathrm{val}(g) = a$ then $\mathrm{val}(f+g)$ can be any value $\geq a$.

$$\mathrm{val}(p^m + p^m) = m$$

$$\mathrm{val}(p^m + (p^n - p^m)) = n \qquad (n \geq m)$$

In general,

$$a \boxplus b = \begin{cases} \{a\} & \text{if } a<b \\ \{b\} & \text{if } b<a \\ [a,\infty] & \text{if } a=b \end{cases}$$

No mention of K

Here $a \boxplus b \ni \infty$ iff $a=b$ so "$a=-a$."

More generally:

$\hat{\boxplus}_{i=1} a_i \ni \infty$ iff $\min\{a_1,\dots,a_n\}$ is achieved at least twice.

See "Hyperfields for Tropical Geometry" by Oleg Viro (2010)

1006.3034

Morphisms to the tropical hyperfield

If R is a commutative ring, a morphism $\phi: R \to \mathbb{T}$ is a map s.t.

$$\phi(a+b) \boxed{\in} \phi(a) \boxplus \phi(b), \qquad \phi(0)=\infty,$$
$$\phi(ab) = \phi(a)\cdot\phi(b), \qquad \phi(1)=0.$$

Theorem: $\mathrm{Mor}(R,\mathbb{T}) = \{(p,v) : p\in \operatorname{Spec} R,\ v \text{ a val. on } \operatorname{Frac}(R/p)\}$

Sketch: The prime ideal p is $\ker\phi = \{r\in R : \phi(r)=\infty\}$.

Note $f\in\ker\phi$, $r\in R \Rightarrow \phi(rf) = \phi(r)\cdot\phi(f) = \infty$.

$f,g\in\ker\phi \Rightarrow \phi(f+g) \in \phi(f)\boxplus\phi(g) = \{\infty\}$.

if $\phi(fg) = \phi(f)\cdot\phi(g) = \infty$ then at least one of $\phi(f),\phi(g)$ is $=\infty$.

We define $v : R/\mathfrak{p} \to \mathbb{R} \cup \{\infty\}$ by

$$\boxed{v(f+\mathfrak{p}) = \phi(f)} \qquad (\mathbb{T} = \mathbb{R} \cup \{\infty\} \text{ as sets})$$

This is well-defined because if $g \in \mathfrak{p}$ then

$$\phi(f+g) \in \phi(f) \boxplus \phi(g) = \{\phi(f)\}.$$

This is a valuation because

$$v(fg) = \phi(f) \cdot \phi(g) = v(f) + v(g)$$

$$v(f+g) \in \phi(f) \boxplus \phi(g) \Rightarrow v(f+g) \geq \min\{v(f), v(g)\}. \quad //$$

This theorem implies that if $X = \operatorname{Spec} R$ then

$$X^{an} = \{\text{seminorms on } R\} = \operatorname{Mor}(R, \mathbb{T}) = X(\mathbb{T}).$$